Research Proposal

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1. Introduction

Idea of ’distance’ is a pertinent concept that surmounts the whole of graph theory. It is inevitable to a number of concepts of symmetry. My primary interest about ‘distance’ in graphs comes from contribution of Paul Erdos to prime distance graphs and next on a text book on “Distance in Graphs” by Buckley and Harary[1].

It is still open to determine the chromatic number of the graph obtained from the Euclidean plane where the points in the plane corresponds to vertices of the graph and an edge is drawn between two points if they are separated by a distance of one unit. Although many variants of this problem have been studied, only a few non-Euclidean spaces seem to have been seriously considered.

My intention is to study prime distance graphs introduced by Eggleton, Erdos, and Skilton[2-3] in 1985. For any set D of positive integers, the distance graph G(Z, D) has vertex set Z and an edge between integers x and y if and only if $|x - y| \in D$. The prime distance graph G(Z, P) is the distance graph with D = P, the set of all primes. It is known that the chromatic number $\chi(G(Z, P)) = 4$. Research in prime distance graphs has since focused on the chromatic number of G(Z, D) where D is a non-empty proper subset of P.

Alternatively, a graph G is a prime distance graph if there exists a one-to-one labeling of its vertices, $L : V(G) \rightarrow Z$ such that for any two adjacent vertices x and y, the integer $|L(x) - L(y)|$ is prime. Define $L(xy) = |L(x) - L(y)|$. We call L a prime distance labeling of G and hence G is a prime distance graph if and only if there exists a prime distance labeling of G. It is interesting to observe that in a prime distance labeling, the labels on the vertices of G must be distinct, but the labels on the edges need not be. Further, it may happen that L(xy) is a prime with xy not an edge of G. Also these graphs are all non-induced infinite subgraphs of G(Z, P). It is my desire to probe this labeling further for several classes of graphs and find applications.

2. Some Results

Given a graph G consisting of vertices and edges, a (vertex) coloring of G is a function f from the vertices of G to a set whose elements are called colors. A k-coloring of G is a coloring that uses k colors. A proper k-coloring of G is a k-coloring f such that $f(x) \neq f(y)$ whenever x and y are adjacent. The chromatic number $\chi(G)$ of G is the minimum number k such that there exists a proper k-coloring of G. Suppose D is a subset of the metric space of all positive integers Z. The integral distance graph G(Z, D) with distance set D is the graph with vertex set Z, and two vertices x and y are adjacent in G if and only if $|x - y| \in D$.

It has been proved that $\chi(G(R^{2,} \{1\}))$ five colors are necessary and seven colors are sufficient for this coloring. The exact number of colors needed, however, has yet to be determined. Eggleton, Erdos, and Skilton[2] investigated general distance graphs G(S, D) with metric space S and distance set D to determine their chromatic numbers $\chi(G(S, D))$. When the metric space in question is the n-dimensional Euclidean space R^n, it is rather difficult to determine $\chi(G(R^n, D))$ or even $\chi(G(Z^n, D))$ for that matter. The above authors studied, among other things, $\chi(G(R, D))$, where D is either 1) a closed interval of distances, 2) an open interval of distances, or 3) a union of infinitely many closed intervals. Continuing their work, Eggleton, Erdos, and Skilton[4] carefully studied prime distance graphs of the form G(Z, D), where D is a subset of the prime numbers. They determined that $\chi(G(Z, P)) = 4$, where P is the set of all primes. Laison et.al[5] proved that trees, cycles, and bipartite graphs are prime distance graphs, and that Dutch windmill graphs and paper mill graphs are prime distance graphs if and only if the Twin Prime Conjecture and dePolignac's Conjecture are true, respectively. They gave a characterization of 2-odd graphs in terms of edge colorings, and used this characterization to determine which circulant graphs of the form Circ(n, {1, k}) are 2-odd and to prove results on circulant prime distance graphs.

Yegnanarayanan has contributed a number of significant results in this area [7-17]. Yegnanarayanan has almost characterized the class 3 and class 4 prime distance graphs by answering a conjecture raised by Kemnitz and Kolberg depending on the Schinzel's conjecture from prime number theory [1]. Given a subset D of positive integers, an integer distance graph is a graph G(Z, D) with the set Z of integers as vertex set and with an edge joining two vertices u and v if and only if $|u-v| \in D$. He considered the problem of determining the chromatic number χ of certain integer distance graphs G(Z, D)whose distance set D is either 1) a set of (n + 1) positive integers for which the n th power of the last is the sum of the n th powers of the previous terms, or 2) a set of pythagorean quadruples, or 3) a set of pythagorean n-tuples, or 4) a set of square distances, or 5) a set of abundant numbers or deficient numbers or carmichael numbers, or 6) a set of polytopic numbers, or 7) a set of happy numbers or lucky numbers, or 8) a set of Lucas numbers, or 9) a set of Ulam numbers, or 10) a set of weird numbers. Besides finding the chromatic number of a few specific distance graphs we also give useful upper and lower bounds for general cases. Further, we raise some open problems. It is known that $\chi(G(Z, P)) = 4$ where P is a set of prime numbers. It is classified that a subset D of P belongs to class i, if $\chi(G(Z, D)) = I$ for i = 1, 2, 3, 4 We looked at the open problem of characterizing class 4 sets when the distance set D is not only a subset of P but also a set of special class of primes He also considered the open problem of characterizing class three and class four sets when the distance set D is not only a subset of primes P but also a special class of primes like Additive primes, Deletable primes, Wedderburn-Etherington Number primes, Euclid-Mullin sequence primes, Motzkin primes, Catalan primes, Schroder primes, Non-generous primes, Pell primes, Primeval primes, Primes of Binary Quadratic Form, Smarandache-Wellin primes, and Highly Cototient number primes. We also have indicated the membership of a number of special classes of prime numbers in class 2 category [2-6]. He obtained a partial solution to a general open problem of characterizing a class of prime distance graphs. He computed the vertex arboricity of certain prime distance graphs. He introduced the notion of pseudochromatic colouring and obtain certain results concerning circulant graphs and distance graphs.

Very recently in 2018, Yegnanarayanan in [8] has computed the χ of certain distance graphs whose distance set elements are (a) a finite set of Catalan numbers, (b) a finite set of generalized Catalan numbers, (c) a finite set of Hankel transform of a transformed sequence of Catalan numbers. Then while discussing the importance of minimizing interference in wireless networks, he probed how a vertex colouring problem is related to minimizing vertex collisions and signal clashes of the associated interference graph. Then when investigating the χ of certain G(V,D) and graphs with interference, he also computed certain lower and upper bound for χ of any given simple graph in terms of the average degree and Laplacian operator.

3. Application Interests of this proposal

Hierarchical data structures are important for many computing and information science disciplines including data mining, terrain modeling, and image analysis. There are many specialized hierarchical data management systems, but they are not always available. Alternatively, relational databases are far more common and offer superior reliability, scalability, and performance. However, relational databases cannot natively store and manage hierarchical data. Labeling schemes resolve this issue by labeling all nodes with alphanumeric strings that can be safely stored and retrieved from a database. One such scheme uses prime numbers for its labeling purposes, however the performance and space utilization of this method are not optimal and in this proposal it has been interest to develop new methods that are optimal and uses our new prime labeling schemes efficiently.

Real-world information often consists of multiple pieces that can be grouped together. Usually, such an abstract relationship is modelled as a hierarchy, e.g., business organization, family ancestry, or military chain of command. The most common application of a hierarchical model is the file system on any modern operating system. It allows thousands of files to be neatly organized into appropriate folders, subfolders, etc. These hierarchical relationships provide an interesting insight into how information is organized. Therefore, a demand exists for data management systems that can store, retrieve, and search such data.

A major distinction between hierarchical and relational data management is the way each method locates data. Hierarchical systems are best suited for gradual refinement of the search criteria or limiting the search to a specific category, subcategory, etc. Due to their advanced indexing ability, relational database systems excel at searches based on fixed criteria. However, no system can provide both kinds of functionality. In fact, this is why modern operating systems generate a file system index in addition to maintaining all files in a hierarchy. As an alternative, attempts have been made to add the hierarchical functionality to an existing relational database. Since classic approaches are limited in their applications, a lot of research has been directed toward creating a more universal labeling scheme.

The prime number labelling scheme (PNL) labels each node with two numbers: a unique prime number called ‘self-label’ and another number called ‘parent label’. Each parent-label is divisible by all of its ancestors’ self-labels because the label is a product of all ancestor self-labels. Adding a node to such a tree is simple. The self-label is assigned a value of any unused prime number and the parent-label is a product of this prime with a parent-label of the parent-node. This labelling scheme determines the relationship between two nodes by comparing their labels. If self-label of node X divides node Y’s parent-label, then node X is considered to be a parent of node Y. Likewise, all nodes whose parent-labels are divisible by prime P are descendants of the node with P as a self-label. A modulo function can automate this process. Itis not computationally intensive and can quickly operate on very large numbers.

The original prime number labeling scheme does not allow self-labels to be reused. This causes parent-labels to grow exponentially, which limits model capacity, increases model size, and reduces performance. In fact, label size limitations dictate the maximum possible depth and fan out of the model. In this proposal, we propose to confirm this by our research.

4. Research Plan

It is proposed to attempt the following problems in this research proposal.

• Determine the chromatic numbers $\chi(G(Z, D)$ of integral distance graphs G(Z, D) with distance sets of 4 elements in fact, one specific result has already been established by Chen, Chang, and Huang[6]: if n is an odd number of at least 5, then $\chi(G(Z, \{2, 3, n, n+2\}) = 4$.

• Determine the chromatic numbers $\chi(G(Z, D)$ of integral distance graphs G(Z, D) with distance sets of 5 elements, 6 elements, etc. • Analyze integral distance graphs in 2 or more dimensions (i.e., $G(Z^2, D)$, $G(Z^3, D)$, etc.).

Is there a family of graphs which are prime distance graphs if and only if Goldbach's Conjecture is true?

What circulant graphs Circ(n, S), for more general sets S, are prime distance graphs?

By the definition of prime distance graphs, L(xy) may still be prime if xy is not an edge of G. How do the existing results change if we define L(xy) to be prime if and only if uv is an edge of G?

What other families of graphs are prime distance graphs?

Can we classify which planar graphs are prime distance graphs?

Generalization to Rings of Integers. Propose to consider p-adic norm distance graphs in addition to the usual Euclidean distance graphs. From a number theoretic point of view, this approach is quite natural because both classes of norms go hand in hand. We intend to consider distance graphs where Z is replaced by other rings R with suitable norms. Specifically we will consider univariate polynomial rings and rings of integers in number fields. It would be of interest to consider the distance graph $G(F_q[x], D)$ where F_q is a finite field with q elements and D is a (non-archimedian) distance set because for such polynomial rings, all norms are non-archemedian. If R is a ring of integers in a number field, then it is possible for R to have several inequivalent archemedian norms. It would be interesting to study how the structure of the distance graphs would change when one transitions from Z to general rings of integers. It is hoped that by studying such generalizations, light would be shed on the standard distance graph problem on Z.

We propose to employ special primes to produce a reusable prime number labeling scheme (rPNL) that recycles deleted labels and produces a more compact and responsive hierarchical model. We also wish to find the applicability of using prime number labeling schemes of various classes of graphs and figure out the most compact relational database hierarchical solutions.

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