A SURVEY ON p - ADIC NUMBERS

Project Report submitted to Ayya Nadar Janaki Ammal College, Sivakasi,

in partial fulfillment of the requirements for the Degree of

Master of Science

in

MATHEMATICS

By

Ms.A. MANIMEGALAI

(Reg.no.: 19PM13)

Centre for Research and Postgraduate Studies in Mathematics

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Dr. C. Parameswaran, M.Sc., M.Phil., B.Ed., Ph.D.
Associate Professor and Head,
Centre for Research and Post Graduate Studies in Mathematics,
Ayya Nadar Janaki Ammal College (Autonomous),
Sivakasi - 626 124.

Certificate

This is to certify that this project report entitled **"A SURVEY ON p - ADIC NUMBERS"** being submitted by **Ms. A. MANIMEGALAI (Reg. No.: 19PM13)**, final year student of M.Sc. degree course in Mathematics, Ayya Nadar Janaki Ammal College (Autonomous), Sivakasi, affiliated to Madurai Kamaraj University, Madurai, is a bonafide record of work carried out by her under the guidance and supervision of **Dr. J. Kannan**, Assistant Professor, Department of Mathematics, Ayya Nadar Janaki Ammal College (Autonomous), Sivakasi.

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April 2021

Signature of the HOD

(C. Parameswaran)

E-mail: parames65_c@yahoo.com

Dr. J. Kannan, M.Sc., M.Phil., Ph.D.,

Assistant Professor,

Department of Mathematics,

Ayya Nadar Janaki Ammal College (Autonomous),

Sivakasi - 626 124.

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It is further certified that to the best of my knowledge, this project report or any part thereof has not been submitted in this College or elsewhere for the award of any other degree or diploma.

Sivakasi

April 2021

Signature of the Guide

(J. Kannan)

E-mail: jayram.kannan@gmail.com

Ms. A. MANIMEGALAI,
II M.Sc. Mathematics (Reg. No.: 19PM13),
Centre for Research and Post Graduate Studies in Mathematics,
Ayya Nadar Janaki Ammal College (Autonomous),
Sivakasi - 626 124.

Declaration

This is to certify that this project report entitled **"A SURVEY ON p - ADIC NUMBERS"** has been carried out by me in the Centre for Research and Post Graduate Studies in Mathematics, Ayya Nadar Janaki Ammal College (Autonomous), Sivakasi, affiliated to Madurai Kamaraj University, Madurai, in partial fulfillment of the requirements for the award of the degree of Master of Science in Mathematics.

I further declare that this report or any part thereof has not been submitted in this College or elsewhere for the award of any other degree or diploma.

Sivakasi
April 2021

A. Manimegalai

Signature of the Student
(A. MANIMEGALAI)

E-mail: manimegalaiamaravathi@gmail.com Reg. no.: 19PM13

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"Nantri marapathu nanranru"

-Thiruvalluvar.

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A. Manimegalai.

(A. MANIMEGALAI)

CONTENTS

CHAPTER-I

INTRODUCTION

The p-adic numbers were discovered by German Mathematician Kurt Hensel around the end of nineteenth century. About a century later, the theory of p-adic numbers has penetrated into several areas of mathematics including number theory, Algebraic geometry, Algebraic Topology and Analysis.

► After 1980s, many scientists recognized the importance of p-adic numbers and many researches studied on p-adic numbers and its various applications in their works (Arefeva et al., 1991; Rozikov 1998; Khrennikov 1997, 2003 and 2004; Ganikhodjaev and Rozikoiv 2009).

► Recently, The Hensel codes and properties of p-adic numbers are keeping its importance to attract the mathematicians and many other scientists with its applications in various areas.

► In the present Review paper, to take the awareness of the mathematicians on p-adic numbers and its importance, I will use resources mentioned above and then I will give some fundamental notions, theorems and properties of the p-adic numbers. Furthermore, how to construct p-adic numbers field Q_p and some comparisons with real numbers is another issue the present paper deals with.

In what follows, we examine these two aspects in detail.

In this project,

Chapter 1, just speak to introduction to p-adic numbers.

Chapter 2, Some Definitions and Theorems.

Chapter 3, p-adic Numbers.

Chapter 4, conclusion

CHAPTER-II

PRILIMINARIES

Now, that we have a norm defined, we can work towards defining the p-adic norm. As it is known that real numbers $\mathbb{R}$ and p-adic numbers Q_p are obtained by the completion of rational numbers $\mathbb{Q}$. Each of these numbers determine the distance between a point on the number line and origin. Euclidean norm states the distance between origin and a point on the real number line with the absolute value at infinity, $(|.|_\infty)$. In the usual absolute value $(|.|_\infty)$, if we take any prime number p instead of infinity then we call this absolute value "p-adic norm" and denoted by $|.|_p$. Most useful and important of the p-adic metric is "it satisfies the strong triangle inequality." $|\alpha + \beta|_p \leq |\alpha|_p + |\beta|_p$, which is also called non-Archimedean. This metric provided the second type of the completion of rational numbers that is called p-adic numbers field Q_p.
Following proposition is very simple but it is very useful.

Basic Definitions and Theorems

Definition:2.1

Let p be any prime number. For any non-zero integer α, let $\mathbf{ord_p}\alpha$ be the highest power p which divides α,

$$\text{(i.e) The greatest } m \text{ such that } \alpha \equiv 0(\text{mod } p^m).$$

Recall that $a \equiv b(mod\ p) \implies p|(a-b)$.

Example

1. $ord_3(54)$

   $54 = 3^3$ x 2

   Highest power of 3 which divides 54 is 3.

   $\therefore ord_3(54) = 3$

2. $ord_2(97)$

   Here, 97 is a prime number.

   $\therefore ord_2(97) = 0$

Note:2.2

If $\alpha = 0$ then $ord_p\alpha = \infty$.

Note:2.3

These behave in similiar fashion to logarithms:

(i) $ord_p(\alpha\beta) = ord_p(\alpha) + ord_p(\beta)$

(ii) $ord_p(\frac{\alpha}{\beta}) = ord_p(\alpha) - ord_p(\beta)$

Proof:

(i) Let $\alpha = p^x q_1$, (p, q_1)=1, then $ord_p(\alpha) = x$.

Let $\beta = p^y q_2$, (p, q_2)=1, then $ord_p(\beta) = y$.

Now, $\alpha\beta = (p^x q_1)(p^y q_2) = p^{x+y} q_1 q_2, \quad (p, q_1 q_2) = 1$

$$\alpha\beta = p^{x+y} q_1 q_2$$

$$\therefore ord_p(\alpha\beta) = x + y$$

$$= ord_p(\alpha) + ord_p(\beta)$$

$$= ord_p(\alpha) + ord_p(\beta)$$

(ii) Now, $\dfrac{\alpha}{\beta} = \dfrac{p^x q_1}{p^x q_2} = p^{x-y} q_1 q_2^{-1}$

$$ord_p(\frac{\alpha}{\beta}) = x - y$$

$$= ord_p(\alpha) - ord_p(\beta)$$

$$\therefore ord_p(\frac{\alpha}{\beta}) = ord_p(\alpha) - ord_p(\beta)$$

Examples for properties

1. $ord_3(\frac{7}{9}) = ord_3(7) - ord_3(9) = 0 - 2 = -2.$

2. $ord_5(-0.0625) \implies -0.0625 = \frac{-625}{10000}$

$$\implies ord_5(-0.0625) = ord_5(-625) - ord_5(10000)$$

$$= 4 - 4$$

$$= 0$$

Definition:2.4

Let p be a prime. For non-zero $\alpha \in \mathbb{Q}$ we define

$|.|_p : \mathbb{Q} \longrightarrow \mathbb{R}^+ \cup \{0\}$, by

$$|\alpha|_p = \begin{cases} \dfrac{1}{p^{ord_p \alpha}} & \text{if } \alpha \neq 0 \\ 0 & \text{if } \alpha = 0 \end{cases}$$

We call this map the **p − adic absolute value or p − adic norm**

Examples:2.5

1. $|250|_5$

$$= \frac{1}{5^{ord_5(250)}}$$
$$= \frac{1}{5^3} \quad (Since \quad ord_5(250) = 3)$$
$$= \frac{1}{125}$$

$\therefore |250|_5 = \dfrac{1}{125}$

2. $\left|10^9\right|_3$

$$= \frac{1}{3^{ord_3(10^9)}}.$$

Now, $ord_3(10^9) = 9(ord_3(10))$

$= 9(0).$

$= 0.$

$\therefore \left|10^9\right|_3 = \dfrac{1}{3^0} = 1.$

3. $|9!|_3 = \dfrac{1}{3^{ord_3(9!)}}$

Now, $\quad ord_3(9!) = ord_3((9)(8)(7)(6)(5)(4)(3)(2)(1))$

$= ord_3(9) + ord_3(8) + ord_3(7) + ord_3(6) + ord_3(5) +$

$ord_3(4) + ord_3(3) + ord_3(2) + ord_3(1).$

$= 2 + 0 + 0 + 1 + 0 + 0 + 1 + 0 + 0.$

$= 4.$

$\therefore |9!|_3 = \dfrac{1}{3^4} = \dfrac{1}{81}.$

Proposition :2.6

The $p-adic$ absolute value $|.|_p$ is a norm on $\mathbb{Q}$.

1. $|\alpha|_p = 0 \iff \alpha = 0.$
2. $|\alpha\beta|_p = |\alpha|_p.|\beta|_p.$
3. $|\alpha + \beta|_p \leq |\alpha|_p + |\beta|_p.$

Proof

1. Assume that $|\alpha|_p = 0.$

   $$\implies \frac{1}{p^{ord_p\alpha}} = 0 \implies p^{ord_p\alpha} = \infty.$$

   Here p is finite but $p^{ord_p\alpha}$ is ∞.

   The only possible value of α is 0.

   $$\implies p^{ord_p\alpha} = p^{ord_p 0} = p^{\infty} = \infty. \qquad (since\ ord_p(0) = \infty).$$

   $\therefore \alpha = 0.$

   Conversely, Assume that $\alpha = 0.$

   $$|0|_p = \frac{1}{p^{ord_p 0}} = \frac{1}{\infty} = 0. \quad (Since\ ord_p(0) = \infty)$$

   $$\therefore |0|_p = 0$$

2. $$|\alpha\beta|_p = \frac{1}{p^{ord_p(\alpha\beta)}}$$
   $$= \frac{1}{p^{ord_p\alpha + ord_p\beta}}$$
   $$= \frac{1}{p^{ord_p\alpha}.p^{ord_p\beta}}$$
   $$= \frac{1}{p^{ord_p\alpha}}.\frac{1}{p^{ord_p\beta}}$$
   $$= |\alpha|_p \,.\, |\beta|_p$$

$$\therefore |\alpha\beta|_p = |\alpha|_p |\beta|_p .$$

3. $|\alpha+\beta|_p$

If $\alpha = 0$ or $\beta = 0$ or $\alpha+\beta = 0$ then it is trivial.

Assume that α, β, and $\alpha+\beta$ are all non-zero.

Let $\alpha = \frac{a}{b}$ and $\beta = \frac{c}{d}$. Then $\alpha+\beta = \dfrac{ad+bc}{bd}$.

$|\alpha+\beta|_p = \dfrac{1}{p^{ord_p(\alpha+\beta)}}$.

Now, $ord_p(\alpha+\beta) = ord_p\dfrac{ad+bc}{bd}$

$$= ord_p(ad+bc) - ord_p(bd)$$

$$= ord_p(ad+bc) - [ord_p(b) + ord_p(d)]$$

$$\geq \min\ (ord_p ad, ord_p bc) - ord_p b - ord_p d$$

$$= min(ord_p a + ord_p d, ord_p b + ord_p c) - ord_p b - ord_p d.$$

$$= \ min(ord_p(a) \ - \ ord_p(b), ord_p(c) \ - \ ord_p(d))$$

$= min(ord_p\alpha, ord_p\beta)$.

$\therefore |\alpha+\beta|_p = p^{-ord_p(\alpha+\beta)} \leq max(p^{-ord_p\alpha}, p^{-ord_p\beta})$.

$$= max(|\alpha|_p , |\beta|_p)$$

and this is $\leq |\alpha|_p + |\beta|_p$.

Definition:2.7

Let p be a prime. A $\mathbf{p-}$**adic Metric** is a non-empty set $\mathbb{Q}$ together with a function $d_p : \mathbb{Q}\times\mathbb{Q} \longrightarrow \mathbb{R}^+ \cup \{0\}$ defined by,

$$d_p(\alpha,\beta) = |\alpha-\beta|_p .$$

Examples:2.8

1. $\alpha = 1$, $\beta = 26$, $p = 3$.

   $(i.e)$ $|1-26|_3$ (i.e. 3-adic "distance" between 1 and 26)

   $\implies |-25|_3 = \dfrac{1}{3^{ord_3(-25)}}$

   Since $ord_3(-25) = 0$.

$$\therefore |-25|_3 = \frac{1}{3^0} = 1.$$

2. $\alpha = 1$, $\beta = 26$, $p = \infty$. $(i.e)$ $|1-26|_\infty = |-25|_\infty = 25$. (Since $|.|_\infty$ denotes the Usual absolute value.)

3. $\left|\dfrac{63}{550}\right|_p$

   Now, $\dfrac{63}{550} = \dfrac{(7)(3^2)}{(5^2)(2)(11^{-1})} = (2^{-1})(3^2)(5^{-2})(7)(11^{-1})$.

   $\left|\dfrac{63}{550}\right|_2 = \dfrac{1}{2^{-1}} = 2$. $(since\ ord_2(\frac{63}{550}) = -1)$

   $\left|\dfrac{63}{550}\right|_3 = \dfrac{1}{3^2} = \dfrac{1}{9}$. $(since\ ord_3(\frac{63}{550}) = 2)$

   $\left|\dfrac{63}{550}\right|_5 = \dfrac{1}{5^{-2}} = 25$. $(since\ ord_5(\frac{63}{550}) = -2)$

   $\left|\dfrac{63}{550}\right|_7 = \dfrac{1}{7^1} = \dfrac{1}{7}$. $(since\ ord_7(\frac{63}{550}) = 1)$

   $\left|\dfrac{63}{550}\right|_{11} = \dfrac{1}{11^{-1}} = 11$. $(since\ ord_{11}(\frac{63}{550}) = -1)$ $\left|\dfrac{63}{550}\right|_{13} = \dfrac{1}{13^0} = 1$. $(since\ ord_{13}(\frac{63}{550}) = 0)$

$$\therefore \left|\frac{63}{550}\right|_p = \begin{cases} 2 & \text{if } p = 2 \\ \frac{1}{9} & \text{if } p = 3 \\ 25 & \text{if } p = 5 \\ \frac{1}{7} & \text{if } p = 7 \\ 11 & \text{if } p = 11 \\ 1 & \text{if } p \geq 13 \end{cases}$$

Definition:2.9

A **norm** is called **non − Archimedean** if $||\alpha + \beta|| \leq \ max\ (||\alpha||, ||\beta||)$ always holds.

A **metric** is called **non − Archimedean** if $d(\alpha, \beta) \leq \max(\mathrm{d}(\alpha, \gamma), \mathrm{d}(\gamma, \beta))$. This is the **Stronger Property**. In the proof above we showed that $|\alpha + \beta|_p \leq |\alpha|_p + |\beta|_p$. Thus, $|.|_p$ is a non-Archimedean norm on $\mathbb{Q}$.

Definition:2.10

A norm (or) metric which is not non-Archimedean is called **Archimedean**.

An example of an Archimedean norm on $\mathbb{Q}$ is the ordinary absolute value.

Definition:2.11

Let $\mathbb{F}$ be a field with an absolute value $|\ |_P$. Let $a \in \mathbb{F}$ be an element and $r \in \mathbf{R}^+$ be a real number. The open ball of radius r and center a is the set

$$B(a,r) \ = \ \{x \in \mathbb{F} : d_p(x,a) < \ r\} \ = \ \{x \in \mathbb{F} : |x-a|_p < \ r\}.$$

The closed ball of radius r and center a is the set

$$\bar{B} = \ \{x \in \mathbb{F} : d_p(x,a) \leq \ r\} \ = \ \{x \in \mathbb{F} : |x-a|_p \leq \ r\}.$$

Definition:2.12

Cauchy sequence

Let S be the set of sequences $\{\alpha_i\}$ of rational numbers such that given $\epsilon > 0$, there exists an M such that $|\alpha_i - \alpha_j|_p < \epsilon$ if both $i, j > M$.

we call two such cauchy sequence $\{\alpha_i\}$ and $\{\beta_i\}$ equivalent if

$$|\alpha_i - \beta_i|_p \to 0 \text{ as } n \to \infty.$$

Proposition:2.13

In an non-Archimedean norm, All "triangles" are isosceles.

Proof:

Let x, y and z be three elements of our space (the vertices of our "triangle").

The lengths of the sides of the "triangle" are the three distances $d(x, y) = |x - y|$, $d(y, z) = |y - z|$ and $d(x, z) = |x - z|$.

Now, of course,

$$(x - y) + (y - z) = (x - z),$$

so that we can invoke the proposition to show that if $|x - y| \neq |y - z|$, then $|x - z|$ is equal to the bigger of the two.

In either case, two of the "sides" are equal.

Lemma:2.14

Let $\mathbb{F}$ be a field with an absolute value $|\ |_p$. The following are equivalent:

i). $\lim_{n\to\infty} x_n = a$.

ii). Any open set containing a also contains all but finitely many of the x_n.

Proof:

Assume Any open set containing a also contains all but finitely many of the x_n.

Since an open ball B(a, ϵ) centered at a is an open set, all but finitely many x_n will be in the open ball, and so there is an N such that $n \geq N$ implies $|x - a| \ < \ \epsilon$,

$$x_n \to a.$$

Conversely, suppose $x_n \to a$ and let U be an open set containing a.

Since U is open there exists an r such that $B(a, r) \subset U$. Therefore there is an N such that $|x - a| \ < \ r$ for all $n \geq N$.

Hence for all but finitely many n we have $x_n \in B(a, r) \subset U$.

Theorem:2.15 (Ostrowski)

Every non-trivial absolute value on $\mathbb{Q}$ is equivalent to one of the absolute values $|.|_p$, where either p is a prime number or $p = \infty$.

Proof:

Let $|\ |$ be a non-trivial absolute value on $\mathbb{Q}$. We will consider the possible cases.

a). Suppose, first that $|\ |$ is archimedean. We want to show that it is equivalent to the "usual" $(\infty - adic)$ absolute value.

Let m_0 be the least positive integer for which $|m_0| \ > 1$.

Now of course we can find a positive real number λ so that

$$|m_0| \ = \ m_0^{\lambda}.$$

We claim that this λ will do, that is, that it will realize the equivalence between $| \ |$ and $| \ |_\infty$

This means that we want to prove that for every $x \in Q$ we have $|x| = |x|_\infty^\lambda$.

Given the known properties of absolute values, this will follow if we know it for positive integers, that is, if we show that $|m| = m^\lambda$ for any positive integer n,

We know that the equality holds for $m = m_0$. To prove it in general, we use a little trick. Take an arbitrary integer m, and write "in base m_0", $(i.e)$, in the form

$$m = a_0 + a_1 m_0 + a_2 m_0^2 + ... + a_k m_0^k,$$

with $0 \leq a_i \leq m_0 - 1$ and $a_k \neq 0$. Notice that k is determined by the inequality $m_0^k \leq m < m_0^{k+1}$, which says that

$$k = \left\lfloor \frac{log\ m}{log\ m_0} \right\rfloor$$

Where $\lfloor x \rfloor$ denotes the "floor" of x, that is, the largest integer that is less than or equal to x. Now take absolute values. We get

$$\begin{aligned} |m| &= \left|a_0 + a_1 m_0 + a_2 m_0^2 + ... + a_k m_0^k\right| \\ &\leq |a_0| + |a_1| m_0^\lambda + ... + |a_k| m_0^{k\lambda}. \end{aligned}$$

Since we chose m_0 to be the smallest integer whose absolute value was greater than 1, we know that $|a_i| \leq 1$, so we get

$$|m| \leq 1 + m_0^\lambda + ... + m_0^{k\lambda} = m_0^{k\lambda} \left(1 + m_0^{-\lambda} + ... + m_0^{-k\lambda}\right)$$

$$= m_0^{k\lambda} \ \sum_{i=0}^{k} m_0^{-i\lambda}$$

$$\leq m_0^{k\lambda} \sum_{i=0}^{\infty} m_0^{-i\lambda}$$

$$= m_0^{k\lambda} \frac{m_0^{\lambda}}{m_0^{\lambda}-1}.$$

If we set $C = \frac{m_0^{\lambda}}{m_0^{\lambda}-1}$. We can read this as saying that

$$|m| \ \leq C m_0^{k\lambda} \leq C m^{\lambda}.$$

Now, we use a drity trick.This formula applies for even m; Applying it to an integer of the form m^N we get

$$\left|m^N\right| \leq C m^{N\lambda}$$

Taking N-th roots, we get

$$|m| \leq \sqrt[N]{C} m^{\lambda}$$

Since any N will do, we can let $N \to \infty$, Which makes $\sqrt[N]{C} \to 1$, and so gives an inequality: $|m| \leq m^{\lambda}$. This is half of what we want.

Now we need to show the inequality in the opposite direction. For that, we go back to the expression in base m_0

$$m \ = a_0 + a_1 m_0 + a_2 m_0^2 + ... + a_k m_0^k.$$

Since $m_0^{k+1} \ > \ m \ \geq m_0^k$, we get

$m_0^{(k+1)\lambda} = \left|m_0^{k+1}\right|$

$$= \left|m + m_0^{k+1} - m\right|$$

$$\leq |m| + \left|m_0^{k+1} - m\right|,$$

where we have made use of the inequality proved in the paragraph.

Now since $m \geq m_0^k$, it follows that

$$|m| \geq m_0^{(k+1)\lambda} - \left(m_0^{k+1} - m_0^k\right)^\lambda$$

$$= m_0^{(k+1)\lambda} \left(1 - \left(1 - \frac{1}{m_0}\right)^\lambda\right)$$

$$= C' m_0^{(k+1)\lambda}$$

$$> C' m^\lambda,$$

and once again $C' = 1 - (1 - 1/m_0)^\lambda$ does not depend on n and is positive.

Using precisely the same trick as before, we get reverse inequality $|m| \geq m^\lambda$, and hence $|m| = m^\lambda$. This proves that $|\ |$ is equivalent to the "usual" absolute value $|\ |_\infty$, as claimed.

b). Now, suppose $|\ |$ is non-archimedean.

Then, as we have shown, we have $|m| \leq 1$ for every integer m.

Since $|\ |$ is non-trivial, there must exist a smallest integer m_0 such that $|m_0| < 1$.

The first thing to see is that m_0 must be a prime number. To see why, suppose that $m_0 = a.b$ with a and b both smaller than m_0. Then, by our choice for m_0, we would have $|a| = |b| = 1$ and $|ab| = |m_0| < 1$, which cannot be. Thus, m_0 is prime, so let's call it by a prime-like name; set $p = m_0$.

Now, of course, we want to show that $|\ |$ is equivalent to the p-adic absolute value, where p is this particular prime,

The next step is to show that if $m \in \mathbb{Z}$ is not divisible by p, then $|m| = 1$.

This is not too hard. If we divide m by p we will have a remainder, so that we can write

$$m = rp + s$$

with $0 < s < p$. By the minimality of p, we have $|s| = 1$.

We also have $|rp| < 1$, because $|r \leq 1$ (because $|\ |$ is non-Archimedean) and $|p| < 1$ (by construction).

Since $|\ |$ is non-archimedean, it follows that $|m| = 1$.

Finally, given any $n \in \mathbb{Z}$, write it as $m = p^{ord_a p}$ so that, $|\ |$ is equivalent to the p-adic absolute value, as claimed.

Norm is equivalent to $|\ |_p$, this concludes the proof of Ostrowski's theorem.

Proposition:2.16

If $\lim_{n\to\infty} x^n = 0$ then $|x|_p < 1$.

Proof

Let $|x|_p < 1$.

Then, $\lim_{n\to\infty} |x^n|_p = 0 \quad (since\, |x^n|_p = |x|_p^n)$

(i.e). $\lim_{n\to\infty} x^n = 0$.

Let us assume that $|x|_p \geq 1$.

Hence we get $|x^n|_p \geq 1 for\ all\ positive\ n$

So, it shows that $\lim_{n\to\infty} x^n \neq 0$.

$\therefore This\ is\ a\ \Rightarrow\Leftarrow with\ \lim_{n\to\infty} x^n = 0$.

$\therefore Then\ we\ Conclude\ that\ the\ proposition\ holds.$

Proposition:2.17

The followings are equivalent

(i). The norm $|.|_p$ is non-Archimedean.

(ii). $|n|_p \leq 1$ for all $n \in Z$.

Proof

$(i) \implies (ii)$

Now, we use induction on n.

Let $n = 1$ then $|1|_p = 1 \leq 1$.

Let $k = n - 1$, then $|k|_p \leq 1$.

Let us show that $|n|_p \leq 1$. We can easily get that, $|n|_p = |n + 1 - 1|_p$

$$|(n-1)+1|_p \leq max\left(|n-1|_p, |1|_p\right)$$
$$= 1.$$

Since $|n - 1|_p = |k|_p \leq 1$ and $|1|_p = 1$.

$\therefore for\ all\ n \in N$, We get $|n|_p \leq 1$.

For all $n \in Z$, $|n|_p \leq 1$ (since $|-n|_p = |n|_p$)

(ii) $\implies$ (i)

$|x + y|_p^n = |(x + y)^n|_p$

$$= \left|\sum_{k=0}^{n} nC_k x^k y^{n-k}\right|_p^{n-k}$$

$$\leq \sum_{k=0}^{n} |nC_k|_p \, |x|_p^k \, |y|_p^{n-k}$$

$$\leq \sum k = 0n \, |x|_p^k \, |y|_p^{n-k}$$

$$\leq (n+1)\left[max\left(|x|_p, |y|_p\right)\right]$$

Then for all $n \in Z$, we obtain that $|x+y|_p \leq \sqrt[n]{n+1} max\left(|x|_p, |y|_p\right)$

Hence as $n \to \infty$, $|x+y|_p \leq max\left(|x|_p, |y|_p\right)$

Since $\lim_{n\to\infty} \sqrt[n]{n+1} = 1$.

Proposition:2.18

Let F be a non-Archimedean field. For all a,x $\in F$, if the inequality $||x-a|| < ||a||$ holds then $||x|| = ||a||$.

Proof

From the strong triangle, we get $||x|| = ||x-a+a|| \leq max\,(||x-a||,\ ||a||) = ||a||$.

On the other hand, it becomes $||a|| = ||a-x+x|| \leq max\,(||x-a||, ||x||)$.

If $||x-a|| > ||x||$ then $||a|| \leq ||x-a||$.

This contradicts with the conditions $||x-a|| < ||a||$.

$\therefore$ it becomes $||x-a|| \leq ||x||$ and $||a|| \leq ||x||$.

Thus $||x|| = ||a||$.

Note:2.19

We can express the proposition above as follows:

Let a, b $\in F$ and $||.||$ is a non-Archimedean norm on the field F. Therefore

$$||a|| \geq ||b|| \implies ||a+b|| \ = \ ||a||.$$

Proposition:2.20

Two equivalent norms on a field F are either both non-Archimedean or both Archimedean.

Proof:

Let $||.||_1$ and $||.||_2$ be any two equivalent norms.

If $||.||_1 \equiv ||.||_2$ then $||x||_1 > 1$ and $||x||_2 > 1$ for any integer x.

Claim: $||.||_1$ $||.||_2$ are either both non-Archimedean or both Archimedean.

To prove this, Let us assume that $||x||_1 > 1$ and $||x||_2 < 1$.

$\therefore$ $||x^n||_1 \to 0$ and $||x^n||_2 \to 0$ as $n \to \infty$.

This show that the sequence (x^n) is a cauchy sequence due to $||.||_1$ but not cauchy due to $||.||_2$.

Which is a contradiction to equivalence of the norms.

Hence, Any norm is either Archimedean or non-Archimedean.

$\therefore$ Two equivalent norms on a field F are either both non-Archimedean or both Archimedean.

Proposition:2.21

If p_1 and p_2 are different primes then $|.|_{p_1}$ is not equivalent to $|.|_{p_2}$.

Proof:

Let $p_1 \neq p_2$ be prime numbers and $|.|_{p_1}$ and $|.|_{p_2}$ be any two norms.
Let $(x_n) = \left(\frac{p_1}{p_2}\right)^n$ be a sequence.
Claim: $|.|_{p_1}$ is not equivalent to $|.|_{p_2}$.

$$|x_n|_{p_1} = \left|\frac{p_1^n}{p_2^n}\right|_{p_1}$$

$$= \frac{1}{p_1^{ord_{p_1}\left(\frac{p_1^n}{p_2^n}\right)}}$$

Now, $ord_{p_1}\left(\frac{p_1^n}{p_2^n}\right) = ord_{p_1}(p_1^n) \; - \; ord_{p_1}(p_2^n)$

$$= \; n \; - \; 0$$

$$= \; n$$

$$\therefore ord_{p_1}\left(\frac{p_1^n}{p_2^n}\right) \; = \; n.$$

$$\Longrightarrow \; \left|\frac{p_1^n}{p_2^n}\right|_{p_1} = \frac{1}{p_1^n} = p_1^{-n}.$$

$\therefore |x_n|_{p_1} = p_1^{-n} \to 0$ as $n \to \infty$.

$$|x_n|_{p_2} = \left|\frac{p_1^n}{p_2^n}\right|_{p_2} = \frac{1}{p_2^{ord_{p_2}\left(\frac{p_1^n}{p_2^n}\right)}}$$

Now, $ord_{p_2}\left(\frac{p_1^n}{p_2^n}\right) = ord_{p_2}(p_1^n) \; - \; ord_{p_2}(p_2^n)$

$$= \; 0 \; - \; n$$

$$= \; -n$$

$\therefore\ ord_{p_2}\left(\frac{p_1^n}{p_2^n}\right) = -n.$

$$\implies \left|\frac{p_1^n}{p_2^n}\right|_{p_2} = \frac{1}{p_2^{-n}} = p_2^n.$$

$\therefore |x_n|_{p_2} = p_2^n \to \infty$ as $n \to \infty$.

Actually for a sequence $(x_n) = \left(\frac{p_1}{p_2}\right)^n$,

$|x_n|_{p_1} \to 0$ but $|x_n|_{p_2} \to \infty$.

Hence, These norms are different.

$\therefore |.|_{p_1}$ is not equivalent to $|.|_{p_2}$.

CHAPTER-III

p - ADIC NUMBERS

In this section we are going to review some important properties of the field of p-adic numbers. Now, we are ready to define the p-adic numbers field Q_p. Let p be a prime number then we define Q_p respect to the norm as a completion of rational numbers $\mathbb{Q}$. We can expand p-adic norm to the p-adic field Q_p.

(i.e) $\mathbb{Q} \subset Q_p$. For all $x \in Q_p$;

Let $g : x \in Q_p \rightarrow ||x||_p$, $f : x \in \mathbb{Q} \rightarrow f(x)$ and $g(x) = f(x)$ then g is called the expansion of f to the p-adic field Q_p and p-adic normed space $(Q_p \ , \ | \ . \ |_p)$ is complete.

Q_p is called p-adic numbers field. Elements of Q_p are equivalence classes of cauchy sequences respect to p-adic norm.

Definition:3.1

Let p be the prime. The $\mathbf{p-adic\ numbers}$ are series of the form,

$$X_{-m}p^{-m} + X_{-m+1}p^{-m+1} + ... + X_0 + X_1p + X_2p^2 + ...$$

where $x_i \in \{0, 1, 2, ...p-1\}$ The set of p-adic numbers is denoted by Q_p.

Example:

$\sqrt{2} \in Q_7$

Here $X_i = \{0, 1, 2, 3, 4, 5, 6\}$ and $p = 7$

$\therefore$ 7-adic expansion of $\sqrt{2}$ is,

$$\sqrt{2} = \ 3 + 1.7 + 2.7^2 + 6.7^3 + 1.7^4 + ...$$

Definition:3.2

A p-adic number $\alpha \in Q_p$ is said to be a $\mathbf{p-adic\ integer}$ if this expansion contains only non negative power of p.

The set of p-adic integers is denoted by Z_p. we have

$$Z_p = \{\alpha \in Q_p : |\alpha|_p \leq 1\}.$$

Example:

1. $5\ in\ Q_2$

   Here $X_i = \{0, 1\}$ and $p = 2$

   $\therefore$ 2-adic expansion of 5 is,

$$5 = 1 + 1.2^2$$

2. $35\ in\ Q_3$

   Here $X_i = \{0, 1, 2\}$ and $p = 3$

   $\therefore$ 3-adic expansion of 35 is,

$$35 = 2 + 2.3 + 0.3^2 + 1.3^3.$$

Definition:3.3

A p-adic integer $\alpha \in Z_p$ is said to be $\mathbf{p-adi\ unit}$ if this expansion whose first digit is non-zero. The set of p-adic unit is denoted by Z_p^*.

$$\text{(i.e) } Z_p^* = \{\alpha \in Z_p : |\alpha|_p = 1\}$$

Example:

$|-13.23|_{11}$

$$|-13.23|_{11} = \frac{1}{11^{ord_{11}(-13.23)}}$$

$$= \frac{1}{11^0} \quad (since\ ord_{11}(-13.23) = 0)$$

$$= \ 1.$$

$\therefore |-13.23|_{11} = 1.$

Completion of $\mathbb{Q}$

Completing $\mathbb{Q}$ a different way:

$\mathbb{Q}$ is not complete with respect to p-adic metric $d_p(\alpha, \beta) = \ |\alpha - \beta|_p$.

Example:

Let $p = 7$. The infinite sum

$$1 + 7 + 7^2 + 7^3 + ...$$

is certainly not element of $\mathbb{Q}$ but sequence.

$$1, \ 1+7, \ 1+7+7^2, \ 1+7+7^2+7^3, ...$$

is a cauchy sequence with respect to the $7-adic$ metric.

Completion of $\mathbb{Q}$ by $|x-y|_p$ gives the field Q_p: The field of p-adic number.

Definition:3.4

The field of p-adic numbers Q_p is a completion of Q with respect to the p-adic metric induced by $| \ . \ |_p$.

we define the set Q_p to be the set of equivalent classes of cauchy sequences.

Theorem:3.5

The field of $p-adic$ numbers, Q_p is complete with respect to $| \ . \ |_p$.

Proof:

Suppose we have a cauchy sequence $\{\alpha_n\}$ in Q_p. Then given an $\epsilon > 0$, there exists an N such that $|\alpha_i - \alpha_j|_p < \epsilon$ when $i, j > N$. Now, let $\epsilon = 1/p^k$, then since $\{\alpha\}$ is cauchy, then $|\alpha_i - \alpha_j|_p < 1/p^k$ which implies,

$$1/p^{ord_p(\alpha_i - \alpha_j)} < 1/p^k$$
$$\implies p^{ord_p(\alpha_i - \alpha_j)} > p^k$$
$$\implies ord_p(\alpha_i - \alpha_j) > k.$$

But this implies that the first k terms of α_i and α_j are the same, and this is the same as making the sequence terms arbitrarily close to some $p-adic\ number$. So any sequence in Q_p that is cauchy with respect to $|.|_p$ will converge. Thus, the field Q_p is Complete.

Lemma:3.6

Let $\alpha \in \{0, 1, 2, ..., p^i - 1\}$, If $x \in \mathbb{Q}$ and $|x|_p \leq 1$ then for all i there exists only one $\alpha \in \mathbb{Z}$ that the inequality $|\alpha - x|_p \leq p^{-i}$ holds.

Proof:

Let $x = \frac{a}{b}$ where $(a, b) = 1$.

Since $|x|_p \leq 1$, p does not divide b, p^i are relatively prime. Hence, there exists any two integers m,n that $mb + np^i \equiv 1(mod\ p)$ holds.

Let $\alpha = a.m$, then we get

$$|\alpha - x|_p = \left|am - \frac{a}{b}\right|_p$$

$$= \left|\frac{a}{b}\right|_p |mb - 1|_p$$

$$\leq |mb - 1|_p$$

$$= \left|np^i\right|_p$$

$$= |n|_p p^{-i}$$

$$\leq p^{-i}.$$

Finally, impending $|\alpha - x|_p \leq p^{-i}$, we can add a multiple of p^i to α, to obtain an integer between 0 and p^i and still the inequality $|\alpha - x|_p \leq p^{-i}$ holds.

Theorem:3.7

In Q_p, there exists one only one (α_i) Cauchy sequence presentation that satisfies the following conditions:

i). $0 \leq \alpha_i \leq p^i - 1$, $\alpha_i \in \mathbb{Z}$, for $i = 1, 2, 3, ...,$

ii). $\alpha_i \equiv \alpha_{i+1}$ (mod p^i), for all $i = 1, 2, 3, ..$

For each equivalence class of α which holds the inequality, $|\alpha|_p \leq 1$.

Proof:

Let (β_i) be a cauchy sequence that shows the equivalence class of α.

Here our aim is finding a sequence (α_i) which satisfies (i) and (ii) and equivalent to the sequence (β_i).

We can ignore initial terms of sequence since $|\beta_i|_p \to |\alpha|_p \leq 1$ as $i \to \infty$.

Let $|\beta_i|_p \leq 1$ for all i and $|\beta_i - \beta_{i'}|_p \leq p^{-j}$, for all $i, i' \geq N(j)$ holds for all $j = 1, 2, 3, ..., N(j)$ positive integer.

Let us take the sequence $N(j)$ be increasing respect to j.

Hence $N(j) \geq j$. From previous lemma, for $0 \leq \alpha_j < p^j$, we can find the integers α_j as $\left|\alpha_j - \beta_{N(j)}\right|_p \leq \dfrac{1}{p^j}$.

In order let us show $\alpha_j \equiv \alpha_{j+1} \pmod{p^j}$ and $(\beta_j) \sim (\alpha_j)$.

First claim; since

$$|\alpha_{j+1} - \alpha_j|_p = \left|\alpha_{j+1} - \beta_{N(j+1)} + \beta_{N(j+1)} - \beta_{N(j)} - \left(\alpha_j - \beta_{N(j)}\right)\right|_p$$

$$\leq max\left(\left|\alpha_{j+1} - \beta_{N(j+1)}\right|_p, \left|-\beta_{N(j+1)} + \beta_{N(j+1)}\right|_p, \left|\alpha_j - \beta_{N(j)}\right|_p\right)$$

$$\leq max\left(\frac{1}{p^{j+1}}, \frac{1}{p^j}, \frac{1}{p^j}\right)$$

$$= \frac{1}{p^j}.$$

then we obtain $\alpha_j \equiv \alpha_{j+1} \pmod{p^j}$.

To prove the second claim, let us take a j. For all $i \geq N(j)$; We can obtain,

$$\begin{aligned}|\alpha_i - \beta_j|_p &= \left|\alpha_i - \alpha_j + \alpha_j - \beta_{N(j)} - \left(\beta_i - \beta_{N(j)}\right)\right|_p \\ &\leq max\left(|\alpha_i - \alpha_j|_p\,, \left|\alpha_j - \beta_{N(j)}\right|_p, \left|\beta_i - \beta_{N(j)}\right|_p\right). \\ &\leq max\left(\frac{1}{p^j}, \frac{1}{p^j}, \frac{1}{p^j}\right) \\ &= \frac{1}{p^j}.\end{aligned}$$

$\therefore |\alpha_i - \beta_i|_p \to 0$, as $i \to \infty$. This proves the equivalency $(\beta_j) \sim (\alpha_j)$.

Now let us prove the uniqueness :

Let us take a sequence (α_i'), and $\alpha_{i_0} \neq \alpha_{i_0}'$ for some i_0.

Then, $\alpha_{i_0} \neq \alpha_{i_0}'$ (mod p^{i_0}) since $0 < \alpha_{i_0}, \alpha_{i_0}' < p^{i_0}$. Therefore from (ii), for $i > \ i_0$, we obtain $\alpha_i \equiv \alpha_{i_0} \neq \alpha_{i_0}' \equiv \alpha_i'$ (mod p^{i_0}).

(i.e). $\alpha_{i_0} \neq \alpha_{i_0}'$ (mod p^{i_0}).

This concludes that $|\alpha_i - \alpha_i'|_p > \dfrac{1}{p^{i_0}}$ for all $i \geq i_0$.

Hence, this proves that the sequences $(\alpha_i), (\alpha_i')$ are not congruent.

Every $\alpha \in Q_P$ can be represented as $\alpha = X_0 + X_1 p + ... + X_{i-1} p^{i-1}$ where $X_i \in \{0, 1, 2, .., p-1\}$. And this representation is called the canonic form of α and it is unique.

Theorem:3.8

A series converges in Q_p iff its terms approach zero.

Proof:

Suppose that $\{\alpha_i\}$ is a sequence in Q_p such that $|\alpha_i|_p \to 0$ as $i \to \infty$.

Then the partial sums $S_n = \alpha_0 + \alpha_1 + ... + \alpha_n$ are Cauchy since for $n > m$:

$$|S_n - S_m|_p = |\alpha_{m+1} + ... + \alpha_n|_p \leq \ max \ \left(|\alpha_{m+1}|_p, ..., |\alpha_p|_p\right)$$

Which goes to zero and both $n, m \to \infty$. Since Q_p is complete this implies convergence.

Now, suppose we have a series that converges in Q_p, then it is Cauchy.

Given an $\epsilon > 0$ there exists an N such that for $n, m > N$, $|S_n - S_m|_p < \epsilon$.

In particular $|S_{n+1} - S_n|_p = |\alpha_{n+1}|_p < \epsilon$.

Thus as $n \to \infty$, the terms approach zero.

Hence the Proof.

CHAPTER-IV

Solved Problems on Q_p

Problem:4.1

Let us try with $n = 6$ and $p = 7$ (*i.e.*, $\sqrt{6}$ in Q_5?).

Solution:

First, we want to find $X_0, X_1, X_2..., 0 \leq X_i \leq 4$
such that $(X_0 + (X_1 \times 5) + (X_2 \times 5^2) + ...)^2 = 1 + (1 \times 5)$.

Comparing the coefficients of 5^0,

$$X_0^2 \equiv 1 \text{ (mod 5)}.$$

The above congruent has 5 possibles like 0,1,2,3,4.

But the solutions are $X_0 = 1 \ or \ 4$.

Take $X_0 = 1$.

comparing the coefficient of 5 on both sides,

$$2X_1 \times 5 \equiv 1 \times 5(\ mod \ 5^2)$$
$$2X_1 \equiv 1(\ mod \ 5)$$

The above congruent has 5 possible like $0, 1, 2, 3, 4$.

Hence $X_1 = 3$. Which implies

$$\begin{aligned} 1 + (1 \times 5) &\equiv (1 + (3 \times 5) + (X_2 \times 5^2) + ...)^2 \\ &\equiv 1 + (1 \times 5) + (2X_2 \times 5^2)(mod \ 5^3) \end{aligned}$$

Comparing the coefficients of 5^2 on both sides,

$$2X_2 \equiv 0(mod\ 5)$$

Hence $X_2 = \ 0$

Proceeding like this,

$$\therefore \sqrt{6} = 1 + (3 \times 5) + (0 \times 5^2) + (4 \times 5^3) + ...$$

Problem:4.2

$X^2 - 2 = 0$ in Q_7 (i.e., $\sqrt{2}$ in Q_7).

Solution

$\left(X_0 + X_1.7 + X_2.7^2 + ...\right)^2 = 2 + 0.7 + 0.7^2 + ...$

$$\left(X_0 + X_1.7 + X_2.7^2 + ...\right)^2 = 2 \implies X^2 \equiv 2(mod\ 7^k)$$

When k=1, $\quad X_0^2 \equiv 2(mod\ 7)$

There are two solutions $X_0 = 3\ and\ X_0 = 4$.

Choosing $X_0 = 3$.

When k=2, Any solution X_1 to the congruence mod 7^2 must also a solution mod 7,

Hence of the form, $\quad X_1 = X_0 + 7Y \implies X_1 = 3 + 7Y$

satisfies, $X_1^2 \equiv 2(mod\ 7^2)$

$\implies (3 + 7Y)^2 \equiv 2(mod\ 7^2)$

$\implies 9 + 49Y^2 + 42Y - 2 \equiv 0(mod\ 7^2)$

$\implies 7(7Y^2 + 6Y + 1) \equiv 0(mod\ 7^2)$

$\implies 6Y + 1 \equiv 0(mod\ 7)$.

The above congruent has unique solution, $Y = 1$.

$X_1 = 3 + 7 \times 1 = 10 \implies \therefore X_1 = 10.$

Hence of the form, $X_2 = X_1 + 7Y \implies X_1 = 10 + 7Y$

satisfies, $X_2^2 \equiv 2(mod\ 7^3)$

$\implies (10 + 7Y)^2 \equiv 2(mod\ 7^3)$

$\implies 100 + 49Y^2 + 140Y - 2 \equiv 0(mod\ 7^3)$

$\implies 49Y^2 + 140Y + 98 \equiv 0(mod\ 7^3)$

$\implies 7(7Y^2 + 20Y + 14) \equiv 0(mod\ 7^3)$

$\implies (7Y^2 + 20Y + 14) \equiv 0(mod\ 7^2)$

$\implies 20Y \equiv 0(mod\ 7^2)$

The above congruent has unique solution, $Y = 14$.

$X_2 = 10 + 14(7) = 10 + 98 = 108 \implies \therefore X_2 = 108$.

$X_0 = 3 = 3$

$X_1 = 10 = 3 + 1.7$

$X_2 = 108 = 3 + (1.7) + (2.7^2)$.

In general formula, $X_{k+1} \equiv X_k^2 + X_k - 2(mod\ 7^{k+1})$

$$\therefore \sqrt{2} = 3 + (1.7) + (2.7^2) + (6.7^3) + (1.7^4) + ...$$

Problem:4.3

Compute $\frac{1}{7}$ in Q_5

Solution

From definition $5^6 \equiv 1(mod\ 7)$,

$$\frac{(5^6 - 1)}{7} = 2232$$

$$\implies \frac{-1}{7} = \frac{2232}{1-5^6}$$

$$\implies \frac{2 + (1.5) + (4.5^2) + (2.5^3) + (3.5^4)}{1-5^6}$$

$$= 21423(1 + 5^6 + 5^{12} + ...)$$

$$= 214230214230214230...$$

Hence $\dfrac{1}{7} = 330214230214230214...$

Which is a way of writing $3 + 3.5^1 + 0.5^2 + ...$

$\therefore \dfrac{1}{7} = 3 + 3.5^1 + 0.5^2 + ... \in Q_5.$

Problem:4.4

What about $n = 3$ and $p = 7$?

Solution:

$$\left(X_0 + X_1.7 + X_2.7^2 + ...\right)^2 = 3 + 0.7 + 0.7^2 + ...$$
$$\left(X_0 + X_1.7 + X_2.7^2 + ...\right)^2 = 3$$

$$X_0^2 \equiv 3(mod\ 7)$$

$$1^2 = 1,\ 2^2 = 4,\ 3^2 = 9 \equiv 2, 4^2 = 16 \equiv 2, 5^2 = 25 \equiv 4, 6^2 = 36 \equiv 1(mod\ 7)$$

As you can see no squares are equivalent to $3(mod\ 7)$, so $X_0^2 \not\equiv 3(mod\ 7)$, which implies $\sqrt{3} \notin Q_7$.

CHAPTER-V

CONCLUSION

In the present project focuses on the $p-adic$ numbers, some of the properties of them and we gave the definition of the $p-adic$ numbers field and some of its important analysis with proofs.
According to the results obtained in the section, we obtain the following conclusions:

1. In an non-Archimedean norm, All "triangles" are isosceles.
2. If p_1 and p_2 are different primes then $|.|_{p_1}$ is not equivalent to $|.|_{p_2}$.
3. A series converges in Q_p iff its terms approach zero.
4. Q_p is Complete.
5. $\sqrt{6}$ in Q_5.
6. $\sqrt{2}$ in Q_7).
7. $\frac{1}{7}$ in Q_5
8. $\sqrt{3} \notin Q_7$

Moreover, we showed some analogy between real numbers and $p-adic$ numbers and some important differences such as non-Archimedean.

Reference:

1. Neal Koblitz, $p-adic\, Numbers$, $p-adic\, Analysis$, and $Zeta-Function$, Second Edition Springer-Verlag 1984.

2. A. J. Baker, *An Introduction to* $p-adic\, Numbers$ *and* $p-adic\, Analysis$ [4/11/2002].

3. U.A. Rozikov Asia Pacific Mathemtics Newsletter October 2013, Volume 3 No 4.

4. General approach of the root of a $p-adic\ number$ Article in Kecies Mohamed, Zerzaihi Tahar/ Filomat 27:3 (2013), 429-434

5. Xavier Caruso *Computations with* $p-adic\ numbers$. Journees Nationales de Calcul Formel, In press, Les cours du CIRM.