A STUDY ON THE DISTRIBUTION OF PRIME NUMBERS

RESEARCH PROPOSAL

Submitted to
Kalaslingam Academy of Research and Education
(Deemed to be University)

for the application of Ph.D programme
in Mathematics

Submitted by

Anjaly Jose
Application No: 202312212

November 2023

Introduction

Number theory is a branch of mathematics that explores the properties and relationships of integers. It delves into fundamental concepts such as prime numbers, divisibility, and modular arithmetic. Key questions in number theory involve understanding the distribution of primes and solving Diophantine equations, which seek integer solutions to polynomial equations. This ancient field finds applications in cryptography, coding theory, and various computational algorithms.

The distribution of prime numbers is a fundamental topic in number theory. The prime numbers, which are natural numbers greater than 1 that have no positive divisors other than 1 and themselves, exhibit interesting patterns in their distribution. The prime number theorem is a key result in this area. There are 4 primes less than ten and 25 primes less than 100. If we go a bit higher, there are exactly 664,579 primes less than 10 million[2]. So, 25% of the numbers under 100 and around 6.6% of the numbers under 10 million are primes. This leads to the question, is there any particular formula by which we can count the number of primes under a given n. For the purpose of detailed studies in this direction, the number of primes less than or equal to a given number x is denoted by the prime counting function $\pi(x)$. The prime number theorem states that $\pi(x)$ is asymptotically equal to $x/\log x$.

The distribution of primes among the positive integers is rather irregular. As we go along the number line, they pop out in no particular order. Some- times a prime number occur just one number after another prime, sometimes there are huge gaps between two primes. One interesting aspect of the distribution of primes is the occurrence of twin primes. Twin primes are pairs of primes that have a difference of 2, such as (11, 13) or (17, 19). The twin prime conjecture posits that there are infinitely many twin primes, but this remains unproven.

The prime gaps, or the differences between consecutive primes, are also a subject of study. Although there are larger and larger prime gaps as numbers increase, the exact nature of these gaps is not fully understood.

There are several related results concerning the distribution of prime numbers. Understanding the distribution of prime numbers continues to be a central area of research in mathematics, and progress in this field has far-reaching implications in number theory and related disciplines. In this research proposal, we aim to study about more about this phenomenon.

Research problem

A study on the distribution of prime numbers.

Objective of the study

Understanding the distribution of prime numbers, which continues to be a central area of research in mathematics.

Procedure to be adopted for the proposed study

One of the standard method adopted to study the distribution of prime numbers are the Sieve theoretic techniques. The basic of this is the Sieve of Eratosthenes.

The aim is to understand these techniques starting from the basic ones, leading up to the existing novel ideas and tools. Along with this, I wish to learn the related mathematical objects and techniques which will help in solving the problems in this area.

Review of literature

In this section, we briefly review the classical and recent works which have been done in this area.

1. The celebrated **Prime Number Theorem** was first proved independently by Charles Jean de la Valèe Poussin [3] and Jacques Hadamard [2].
2. The question of distribution of primes in arithmetic progression was settled by Dirichlet in 1837 [1].
3. Yitang Zhang announced a proof that for some integer N that is less than 70 million, there are infinitely many pairs of primes that differ by N [4].
4. The sequence of prime numbers contains arbitrarily long arithmetic progressions. In other words, for every natural number k, there exist arithmetic progressions of primes with k terms. This was proved by Ben Green and Terence Tao [5].

The above results were improved significantly and also generalised to other settings later.

Relevance of the proposed study

Research in Number theory has both theoretical and practical applications.

1. Cryptography: Number theory is essential in the field of cryptography, which secures communication and data. Algorithms like RSA (Rivest–Shamir–Adleman) heavily rely on the mathematical properties of prime numbers. Advances in number theory contribute to the development of more secure encryption methods.

2. Computer Science: Number theory has practical applications in computer science, particularly in algorithms and computational theory. Efficient algorithms for prime number generation and factorization, essential in various computational tasks, often draw on number theoretic principles.

3. Coding Theory: In telecommunications and data transmission, coding theory employs concepts from number theory to design error-correcting codes. These codes help ensure accurate data transmission in the presence of noise or errors.

4. Pure Mathematics: Number theory is a foundational branch of pure mathematics, providing insights into the intrinsic properties of integers. Ongoing research in this area deepens our understanding of mathematical structures, contributes to solving open problems, and enriches the broader mathematical landscape.

5. Cryptocurrency and Blockchain: The security protocols in many cryptocurrencies, including Bitcoin, are based on cryptographic principles rooted in number theory. Understanding the mathematical foundations is crucial for ensuring the security and integrity of decentralized systems.

6. Physics and Quantum Mechanics: Certain mathematical structures and ideas from number theory appear in theoretical physics, particularly in areas related to symmetry and the behavior of particles. Advances in number theory can, therefore, have implications for understanding the fundamental nature of the physical universe.

7. Puzzles and Recreational Mathematics: Number theory often provides the tools and concepts for solving mathematical puzzles and engaging in recreational mathematics. This not only contributes to the joy of exploration but also occasionally leads to unexpected applications.

Bibliography

[1] P. G. L. Dirichlet, *Beweis des Satzes, dass jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthält*, Abhandlungen der Königlichen Preußischen Akademie der Wissenschaften zu Berlin, 48: 45–71

[2] Jacques Hadamard, *Sur la distribution des zéros de la fonction* $\zeta(s)$ *et ses conséquences arithmétiques*, Bulletin de la Société Mathématique de France, 24 (1896), p. 199-220

[3] Charles Jean de la Valèe Poussin, *Recherches analytiques sur la tèorie des nombres premiers*, Annales de la Société scientifique de Bruxelles, Imprimeur de l'Acadèmie Royale de Belgique, 20 B, 21 B: 183–256, 281–352, 363–397, 351–368.

[4] Yitang Zhang. *Bounded gaps between primes*, Annals of Mathematics, 179 (3)(2014): 1121–1174.

[5] Ben Green and Terence Tao, *The primes contain arbitrarily long arithmetic progressions*, Annals of Mathematics. 167 (2) (2008): 481–547.